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On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}$

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ABSTRACT

In this work we investigate the global behavior of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}, \quad n = 0, 1, 2, \ldots$$

where $\alpha \in (0, \infty)$ and $k \in (0, \infty)$.

1. Introduction and preliminaries

The global stability, the boundedness character and the periodic nature of the recursive sequence

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k} \tag{1}$$

were investigated for $k = 1$ where $\alpha \in (0, \infty)$; see Amleh, Grove, and Ladas [1], Feuer [2].

In this work we study the case when $\alpha \in (0, \infty)$, $k \in (0, \infty)$ and the initial conditions x_{-1} and x_0 are arbitrary positive real numbers.

The results in this work are consistent with the results in [1] and [2] when $k = 1$. Some related results are found in [3,4]. For definitions and notation we refer the reader to [5–8].

Let $I \subseteq \mathbb{R}$ be an interval, bounded or not, and let

$$f : I \times I \longrightarrow I$$

be a continuously differentiable function. For every set of initial conditions $\{x_{-1}, x_0\} \subset I$ the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, 2, \ldots \tag{2}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

Let $\bar{x}$ be an equilibrium point of Eq. (2), i.e. $\bar{x} = f(\bar{x}, \bar{x})$. If we replace x_n and x_{n-1} in Eq. (2) by the variables u and v respectively, then we have

$$p = \frac{\partial f}{\partial u}(\bar{x}, \bar{x})$$
$$q = \frac{\partial f}{\partial v}(\bar{x}, \bar{x}).$$

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The equation

$$y_{n+1} = py_n + qy_{n-1}, \quad n = 0, 1, 2, \ldots \tag{3}$$

is called the linearized equation associated with Eq. (2) about the equilibrium point $\bar{x}$. Its characteristic equation is

$$\lambda^2 - p\lambda - q = 0. \tag{4}$$

Theorem 1.1. *[[7,8]]*

(i) *If both roots of the quadratic equation (4) lie in the open disk $|\lambda| < 1$ then the equilibrium point $\bar{x}$ of Eq. (2) is locally asymptotically stable.*

(ii) *A necessary and sufficient condition for both roots of Eq. (4) to lie in the open disk $|\lambda| < 1$ is*

$$|p| < 1 - q < 2.$$

In this case $\bar{x}$ is locally asymptotically stable.

(iii) *A necessary and sufficient condition for both roots of Eq. (4) to lie outside the open disk $|\lambda| < 1$ is*

$$|q| > 1 \quad \text{and} \quad |p| < |1 - q|.$$

In this case $\bar{x}$ is unstable and called a repeller point.

(iv) *A necessary and sufficient condition for one of the roots of Eq. (4) to lie outside the open disk $|\lambda| < 1$ and the other root inside it is*

$$p^2 + 4q > 0 \quad \text{and} \quad |p| > |1 - q|.$$

In this case $\bar{x}$ is unstable and called a saddle point.

(v) *A necessary and sufficient condition that a root of Eq. (4) has absolute value equal to 1 is*

$$|p| = |1 - q|$$

or

$$q = -1 \quad \text{and} \quad |p| \leq 2.$$

In this case $\bar{x}$ is called a non-hyperbolic point.

2. The recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}$

In this section we study the global behavior of

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}, \quad n = 0, 1, 2, \ldots \tag{5}$$

where x_{-1}, x_0 are positive numbers and α, k are positive real numbers.

A point $\bar{x} \in \mathbb{R}$ is an equilibrium point of Eq. (5) if and only if it is a root for the function

$$g(x) = x - x^{1-k} - \alpha, \tag{6}$$

that is

$$\bar{x} - \bar{x}^{1-k} - \alpha = 0. \tag{7}$$

Or equivalently it satisfies the relation

$$\bar{x}\left(1 - \frac{1}{\bar{x}^k}\right) = \alpha. \tag{8}$$

Lemma 2.1. *The following statements are true:*

(i) *If $k = 1$ then Eq. (5) has a unique equilibrium point $\bar{x} = \alpha + 1$.*

(ii) *If $k \neq 1$ then Eq. (5) has a unique equilibrium point $\bar{x} > 1$.*

Proof. (i) If $k = 1$, Eq. (7) gives $\bar{x} = 1 + \alpha$; see [1].

(ii) If $k \neq 1$ we have the following cases.

Case (1) $(0 < k < 1)$.

The function g defined by Eq. (6) is decreasing on $[0, (1-k)^{\frac{1}{k}}]$ and increasing on $[(1-k)^{\frac{1}{k}}, \infty[$. In view of $g(1) = -\alpha < 0$ and $\lim_{x\to\infty} g(x) = \infty$, then g has a unique root $\bar{x} > 1$.

Case (2) $(1 < k)$.

Since g is increasing on $[0, \infty[$, $g(1) = -\alpha < 0$ and $\lim_{x\to\infty} g(x) = \infty$, then g has a unique equilibrium point $\bar{x} > 1$. $\square$

Now we summarize the main results of this section in the following theorem.

Theorem 2.2. *Let $\bar{x}$ be the equilibrium point of Eq. (5).*

(i) *If α satisfies the condition*

$$k(1+k)^{\frac{1-k}{k}} < \alpha, \tag{9}$$

then $\bar{x}$ is locally asymptotically stable.

(ii) *If α satisfies the condition*

$$k(1+k)^{\frac{1-k}{k}} > \alpha, \tag{10}$$

then $\bar{x}$ is unstable, in fact a saddle point.

(iii) *If α satisfies the condition*

$$\alpha = k(1+k)^{\frac{1-k}{k}} \tag{11}$$

then $\bar{x}$ is a non-hyperbolic point.

Proof. From Eqs. (1), (2) and in view of Theorem 1.1 we see that $f(u, v) = \alpha + u^{-k}v$; then

$$p = \frac{-k}{\bar{x}^k}$$
$$q = \frac{1}{\bar{x}^k}. \tag{12}$$

(i) By using Theorem 1.1 we get that $\bar{x}$ is locally asymptotically stable if and only if $\bar{x}^k > k + 1$. A simple calculation, using condition (9), shows that $g((k+1)^{\frac{1}{k}}) = k(k+1)^{\frac{1-k}{k}} - \alpha < 0$ where $g(x)$ is defined by Eq. (6). Then since $\lim_{x\to\infty} g(x) = \infty$, $\bar{x} > (k+1)^{\frac{1}{k}}$.

(ii) The condition $p^2 + 4q > 0$ of Theorem 1.1 is always satisfied and so $\bar{x}$ is unstable if $\bar{x}^k < (k + 1)$.
By condition (10), we have $g((k+1)^{\frac{1}{k}}) = k(k+1)^{\frac{1-k}{k}} - \alpha > 0$. Then since $g(0) < 0$, $\bar{x} < (k+1)^{\frac{1}{k}}$.

(iii) The condition $|p| = |1 - q|$ is equivalent to $\bar{x} = (k+1)^{\frac{1}{k}}$. Similarly by condition (11), we have $g((k+1)^{\frac{1}{k}}) = 0$. Then $\bar{x} = (k+1)^{\frac{1}{k}}$. □

The results of Theorem 2.2 are consistent with the results in [1], when $k = 1$.

Lemma 2.3. *If $\alpha \neq 1$, then every solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (5) satisfies the following inequalities:*

$$\alpha < x_{2n} < \alpha\frac{1-\beta^n}{1-\beta} + \beta^n x_0, \quad n = 1, 2, \ldots$$
$$\alpha < x_{2n+1} < \alpha\frac{1-\beta^n}{1-\beta} + \beta^n x_1, \quad n = 1, 2, \ldots. \tag{13}$$

Here $\beta = \frac{1}{\alpha^k}$.

Proof. We have $\alpha < x_{n+1} < \alpha + \beta x_{n-1} \forall n = 1, 2, \ldots$. By induction we obtain inequalities (13). □

In view of $\alpha > 1$, we also see that

$$\alpha < x_{2n} < \frac{\alpha}{1-\beta} + x_0, \quad n = 1, 2, \ldots$$
$$\alpha < x_{2n+1} < \frac{\alpha}{1-\beta} + x_1, \quad n = 1, 2, \ldots.$$

Theorem 2.4. *Let $\bar{x}$ be the equilibrium point of Eq. (5); if $\alpha > k^{\frac{1}{k}} \geq 1$ then $\bar{x}$ is globally asymptotically stable.*

Proof. First since $k \geq 1$, then

$$k^{\frac{1}{k}} \geq k(1+k)^{\frac{1-k}{k}}$$

and in view of $\alpha > k^{\frac{1}{k}}$ we have $\alpha > k(1+k)^{\frac{1-k}{k}}$. By Theorem 2.2, $\bar{x}$ is locally asymptotically stable.
Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (5). By Lemma 2.3, $\{x_n\}_{n=-1}^{\infty}$ is bounded. We define

$$\lambda = \liminf x_n$$
$$\Lambda = \limsup x_n.$$

Then for all $\varepsilon \in (0, \lambda)$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\lambda - \varepsilon < x_n < \Lambda + \varepsilon.$$

This implies that

$$\alpha + \frac{\lambda - \varepsilon}{(\Lambda + \varepsilon)^k} < x_{n+1} < \alpha + \frac{\Lambda + \varepsilon}{(\lambda - \varepsilon)^k} \quad \forall n \geq n_0 + 1,$$

and then

$$\alpha + \frac{\lambda - \varepsilon}{(\Lambda + \varepsilon)^k} < \lambda \leq \Lambda < \alpha + \frac{\Lambda + \varepsilon}{(\lambda - \varepsilon)^k}. \tag{14}$$

From inequality (14) we conclude that

$$\alpha + \frac{\lambda}{\Lambda^k} \leq \lambda \leq \Lambda \leq \alpha + \frac{\Lambda}{\lambda^k}.$$

From which we get

$$(\alpha \Lambda^k \lambda^{k-1} + \lambda^k) \leq (\Lambda^k \lambda^k) \leq (\alpha \Lambda^{k-1} \lambda^k + \Lambda^k).$$

Consequently, we obtain $(\alpha \Lambda^{k-1} \lambda^{k-1})(\Lambda - \lambda) \leq (\Lambda^k - \lambda^k)$. Supposing that $\Lambda \neq \lambda$ $(\Lambda > \lambda)$ we get

$$\alpha \Lambda^{k-1} \lambda^{k-1} \leq \frac{\Lambda^k - \lambda^k}{\Lambda - \lambda}.$$

There exists $C \in (\lambda, \Lambda)$ such that

$$\frac{\Lambda^k - \lambda^k}{\Lambda - \lambda} = kC^{k-1} \leq k\Lambda^{k-1}.$$

This implies that $\alpha^k \leq k$, which is a contradiction. □

3. Semicycle analysis

Theorem 3.1. *Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (5) which consists of at least two semicycles. Then $\{x_n\}_{n=-1}^{\infty}$ is oscillatory and, except possibly for the first semicycle, every semicycle is of length* 1.

Proof. Assume that $x_{n-1} < \bar{x} \leq x_n$ for some $n \geq 0$; then

$$x_{n+1} < \alpha + \frac{\bar{x}}{\bar{x}^k} = \bar{x},$$

and

$$x_{n+2} > \alpha + \frac{\bar{x}}{\bar{x}^k} = \bar{x}.$$

Second, consider $x_n < \bar{x} \leq x_{n-1}$; then

$$x_{n+1} > \alpha + \frac{\bar{x}}{\bar{x}^k} = \bar{x},$$

and

$$x_{n+2} < \alpha + \frac{\bar{x}}{\bar{x}^k} = \bar{x},$$

which ends the proof. □

4. Existence of period 2 solutions

Theorem 4.1. *Eq. (5) has a period 2 solution (not necessary prime) $\{x_n\}_{n=-1}^{\infty}$ if and only if (x_{-1}, x_0) is a solution of the system*

$$\begin{aligned} x &= \alpha + \frac{x}{y^k} \\ y &= \alpha + \frac{y}{x^k}. \end{aligned} \tag{15}$$

Furthermore if $x_{-1} \neq x_0$, then $\{x_n\}_{n=-1}^{\infty}$ is a prime period 2 solution.

Proof. First, let $\{x_n\}_{n=-1}^{\infty}$ be a period 2 solution of Eq. (5); then

$$x_{-1} = x_1 = \alpha + \frac{x_{-1}}{x_0^k}$$

and

$$x_0 = x_2 = \alpha + \frac{x_0}{x_1^k} = \alpha + \frac{x_0}{x_{-1}^k}.$$

Then (x_{-1}, x_0) is a solution of system (15).

Second, let (x_{-1}, x_0) be a solution of system (15); then

$$x_1 = \alpha + \frac{x_{-1}}{x_0^k} = x_{-1}$$

and

$$x_2 = \alpha + \frac{x_0}{x_1^k} = \alpha + \frac{x_0}{x_{-1}^k} = x_0.$$

By induction we see that

$$x_{n+2} = x_n \quad \forall n \geq -1. \tag{16}$$

In the case where $x_{-1} \neq x_0$, clearly $\{x_n\}_{n=-1}^{\infty}$ is a prime period 2 solution. □

As a direct consequence of Theorem 4.1, we obtain the following results for $k \in \mathbb{N}$.

Corollary 4.2. (i) *Assume that $\alpha = 1$. Then Eq. (5) has a prime period 2 solution iff $k = 1$.*
(ii) *Assume that $k = 1$. Then Eq. (5) has a prime period 2 solution iff $\alpha = 1$.*

Proof. (i) Assume that $\alpha = 1$ and Eq. (5) has a prime period 2 solution. Hence by Theorem 4.1, system (15) has a solution (x, y), such that $x \neq y$ and $x, y \neq 1$. Suppose on the contrary that $k \in \{2, 3, \ldots\}$. Then we have

$$xy^k = y^k + x$$
$$yx^k = x^k + y.$$

Hence,

$$xy^k - yx^k = y^k - x^k - (y - x)$$
$$xy(y^{k-1} - x^{k-1}) = (y^k - x^k) - (y - x)$$
$$xy(y^{k-2} + y^{k-3}x + \cdots + yx^{k-3} + x^{k-2}) = y^{k-1} + y^{k-2}x + \cdots + yx^{k-2} + x^{k-1} - 1$$
$$y^{k-1}(x-1) + y^{k-2}x(x-1) + \cdots + yx^{k-2}(x-1) = x^{k-1} - 1.$$

Since $x \neq 1$, then

$$y^{k-1} + y^{k-2}x + \cdots + yx^{k-2} = x^{k-2} + x^{k-3} + \cdots + 1$$
$$(y^{k-1} - 1) + (y^{k-2} - 1)x + \cdots + (y - 1)x^{k-2} = 0.$$

And since $y \neq 1$,

$$(y^{k-2} + y^{k-3} + \cdots + y + 1) + (y^{k-3} + y^{k-4} + \cdots + y + 1)x + \cdots + x^{k-2} = 0, \quad k \geq 2$$

which is a contradiction.

Conversely, let $k = 1$. Then $(x, \frac{x}{x-1})$ is a solution of system (15) $\forall x \neq 1$ and every solution $\{x_n\}_{n=-1}^{\infty}$ with initial conditions x_{-1}, x_0 such that $x_0 = \frac{x_{-1}}{x_{-1}-1}$, $x_{-1} \neq 1$ is a prime period 2 solution.

(ii) Assume that $k = 1$ and Eq. (5) has a prime period 2 solution $\{x_n\}_{n=-1}^{\infty}$. Then (x_{-1}, x_0) is a solution of system (15) and $x_{-1} \neq x_0$. Hence, $\alpha(y - x) - (y - x) = 0$ which means that $\alpha = 1$.

Conversely, if $\alpha = 1$, then $(x, \frac{x}{x-1})$ is a solution of system (15), $\forall x \neq 1$. □

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